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4. Cohomological symbol for henselian discrete valuation fields of mixed characteristic

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4.1. Cohomological symbol map

Let K be a field. If m is prime to the characteristic of K, there exists an isomorphism

$$h_{1,K}: K^*/m \to H^1(K, \mu_m)$$

supplied by Kummer theory. Taking the cup product we get

$$(K^*/m)^q \to H^q(K, \mathbb{Z}/m(q))$$

and this factors through (by [T])

$$h_{q,K}: K_q(K)/m \to H^q(K, \mathbb{Z}/m(q)).$$

This is called the cohomological symbol or norm residue homomorphism.

Milnor–Bloch–Kato Conjecture. *For every field K and every positive integer m which is prime to the characteristic of K the homomorphism $h_{q,K}$ is an isomorphism.*

This conjecture is shown to be true in the following cases:

(i) K is an algebraic number field or a function field of one variable over a finite field and $q = 2$, by Tate [T].
(ii) Arbitrary K and $q = 2$, by Merkur'ev and Suslin [MS1].
(iii) $q = 3$ and m is a power of 2, by Rost [R], independently by Merkur'ev and Suslin [MS2].
(iv) K is a henselian discrete valuation field of mixed characteristic $(0, p)$ and m is a power of p, by Bloch and Kato [BK].
(v) (K, q) arbitrary and m is a power of 2, by Voevodsky [V].

For higher dimensional local fields theory Bloch–Kato's theorem is very important and the aim of this text is to review its proof.

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Theorem (Bloch–Kato)**.** *Let K be a henselian discrete valuation fields of mixed characteristic $(0,p)$ (i.e., the characteristic of K is zero and that of the residue field of K is $p>0$), then*

$$h_{q,K}\colon K_q(K)/p^n \longrightarrow H^q(K,\mathbb{Z}/p^n(q))$$

is an isomorphism for all n.

Till the end of this section let K be as above, $k=k_K$ the residue field of K.

4.2. Filtration on $K_q(K)$

Fix a prime element π of K.

Definition.

$$U_m K_q(K) = \begin{cases} K_q(K), & m=0 \\ \langle\{1+\mathcal{M}_K^m\}\cdot K_{q-1}(K)\rangle, & m>0. \end{cases}$$

Put $\mathrm{gr}_m K_q(K) = U_m K_q(K)/U_{m+1}K_q(K)$.

Then we get an isomorphism by [FV, Ch. IX sect. 2]

$$K_q(k)\oplus K_{q-1}(k) \xrightarrow{\rho_0} \mathrm{gr}_0 K_q(K)$$
$$\rho_0\left(\{x_1,\dots,x_q\},\{y_1,\dots,y_{q-1}\}\right) = \{\widetilde{x_1},\dots,\widetilde{x_q}\}+\{\widetilde{y_1},\dots,\widetilde{y_{q-1}},\pi\}$$

where $\widetilde{x}$ is a lifting of x. This map ρ_0 depends on the choice of a prime element π of K.

For $m\geqslant 1$ there is a surjection

$$\Omega_k^{q-1}\oplus\Omega_k^{q-2} \xrightarrow{\rho_m} \mathrm{gr}_m K_q(K)$$

defined by

$$\left(x\frac{dy_1}{y_1}\wedge\dots\wedge\frac{dy_{q-1}}{y_{q-1}},0\right)\longmapsto\{1+\pi^m\widetilde{x},\widetilde{y_1},\dots,\widetilde{y_{q-1}}\},$$
$$\left(0,x\frac{dy_1}{y_1}\wedge\dots\wedge\frac{dy_{q-2}}{y_{q-2}}\right)\longmapsto\{1+\pi^m\widetilde{x},\widetilde{y_1},\dots,\widetilde{y_{q-2}},\pi\}.$$

Definition.

$$k_q(K) = K_q(K)/p,\ h_q(K) = H^q(K,\mathbb{Z}/p(q)),$$
$$U_m k_q(K) = \mathrm{im}(U_m K_q(K)) \text{ in } k_q(K), \qquad U_m h^q(K) = h_{q,K}(U_m k_q(K)),$$
$$\mathrm{gr}_m h^q(K) = U_m h^q(K)/U_{m+1}h^q(K).$$

□

Proposition. *Denote* $\nu_q(k) = \ker(\Omega_k^q \xrightarrow{1-\mathrm{C}^{-1}} \Omega_k^q / d\Omega_k^{q-1})$ *where* C^{-1} *is the inverse Cartier operator:*

$$x\frac{dy_1}{y_1} \wedge \dots \wedge \frac{dy_q}{y_q} \longmapsto x^p \frac{dy_1}{y_1} \wedge \dots \wedge \frac{dy_q}{y_q}.$$

Put $e' = pe/(p-1)$*, where* $e = v_K(p)$*.*

(i) *There exist isomorphisms* $\nu_q(k) \to k_q(k)$ *for any* q*; and the composite map denoted by* $\widetilde{\rho}_0$

$$\widetilde{\rho}_0 \colon \nu_q(k) \oplus \nu_{q-1}(k) \xrightarrow{\sim} k_q(k) \oplus k_{q-1}(k) \xrightarrow{\sim} \mathrm{gr}_0 k_q(K)$$

is also an isomorphism.

(ii) *If* $1 \leqslant m < e'$ *and* $p \nmid m$*, then* ρ_m *induces a surjection*

$$\widetilde{\rho}_m \colon \Omega_k^{q-1} \to \mathrm{gr}_m k_q(K).$$

(iii) *If* $1 \leqslant m < e'$ *and* $p \mid m$*, then* ρ_m *factors through*

$$\widetilde{\rho}_m \colon \Omega_k^{q-1}/Z_1^{q-1} \oplus \Omega_k^{q-2}/Z_1^{q-2} \to \mathrm{gr}_m k_q(K)$$

and $\widetilde{\rho}_m$ *is a surjection. Here we denote* $Z_1^q = Z_1\Omega_k^q = \ker(d\colon \Omega_k^q \to \Omega_k^{q+1})$.

(iv) *If* $m = e' \in \mathbb{Z}$*, then* $\rho_{e'}$ *factors through*

$$\widetilde{\rho}_{e'} \colon \Omega_k^{q-1}/(1 + a\,\mathrm{C})Z_1^{q-1} \oplus \Omega_k^{q-2}/(1 + a\,\mathrm{C})Z_1^{q-2} \to \mathrm{gr}_{e'} k_q(K)$$

and $\widetilde{\rho}_{e'}$ *is a surjection.*

Here a *is the residue class of* $p\pi^{-e}$*, and* C *is the Cartier operator*

$$x^p \frac{dy_1}{y_1} \wedge \dots \wedge \frac{dy_q}{y_q} \mapsto x\frac{dy_1}{y_1} \wedge \dots \wedge \frac{dy_q}{y_q}, \quad d\Omega_k^{q-1} \to 0.$$

(v) *If* $m > e'$*, then* $\mathrm{gr}_m k_q(K) = 0$.

Proof. (i) follows from Bloch–Gabber–Kato's theorem (subsection 2.4). The other claims follow from calculations of symbols. □

Definition. Denote the left hand side in the definition of $\widetilde{\rho}_m$ by G_m^q. We denote the composite map $G_m^q \xrightarrow{\widetilde{\rho}_m} \mathrm{gr}_m k_q(K) \xrightarrow{h_{q,K}} \mathrm{gr}_m h^q(K)$ by $\overline{\rho}_m$; the latter is also surjective.

4.3

In this and next section we outline the proof of Bloch–Kato's theorem.

4.3.1. Norm argument.

We may assume $\zeta_p \in K$ to prove Bloch–Kato's theorem.
Indeed, $|K(\zeta_p) : K|$ is a divisor of $p-1$ and therefore is prime to p. There exists a norm homomorphism $N_{L/K}\colon K_q(L) \to K_q(K)$ (see [BT, Sect. 5]) such that the following diagram is commutative:

$$\begin{array}{ccccc}
K_q(K)/p^n & \longrightarrow & K_q(L)/p^n & \xrightarrow{N_{L/K}} & K_q(K)/p^n \\
\big\downarrow{\scriptstyle h_{q,K}} & & \big\downarrow{\scriptstyle h_{q,L}} & & \big\downarrow{\scriptstyle h_{q,K}} \\
H^q(K,\mathbb{Z}/p^n(q)) & \xrightarrow{\text{res}} & H^q(L,\mathbb{Z}/p^n(q)) & \xrightarrow{\text{cor}} & H^q(K,\mathbb{Z}/p^n(q))
\end{array}$$

where the left horizontal arrow of the top row is the natural map, and res (resp. cor) is the restriction (resp. the corestriction). The top row and the bottom row are both multiplication by $|L : K|$, thus they are isomorphisms. Hence the bijectivity of $h_{q,K}$ follows from the bijectivity of $h_{q,L}$ and we may assume $\zeta_p \in K$.

4.3.2. Tate's argument.

To prove Bloch–Kato's theorem we may assume that $n = 1$.
Indeed, consider the cohomological long exact sequence

$$\dots \to H^{q-1}(K,\mathbb{Z}/p(q)) \xrightarrow{\delta} H^q(K,\mathbb{Z}/p^{n-1}(q)) \xrightarrow{p} H^q(K,\mathbb{Z}/p^n(q)) \to \dots$$

which comes from the Bockstein sequence

$$0 \longrightarrow \mathbb{Z}/p^{n-1} \xrightarrow{p} \mathbb{Z}/p^n \xrightarrow{\text{mod } p} \mathbb{Z}/p \longrightarrow 0.$$

We may assume $\zeta_p \in K$, so $H^{q-1}(K,\mathbb{Z}/p(q)) \simeq h_{q-1}(K)$ and the following diagram is commutative (cf. [T, §2]):

$$\begin{array}{ccccccc}
k_{q-1}(K) & \xrightarrow{\{*,\zeta_p\}} & K_q(K)/p^{n-1} & \xrightarrow{p} & K_q(K)/p^n & \xrightarrow{\text{mod } p} & k_q(K) \\
\big\downarrow{\scriptstyle h_{q-1,K}} & & {\scriptstyle h_{q,K}}\big\downarrow & & {\scriptstyle h_{q,K}}\big\downarrow & & {\scriptstyle h_{q,K}}\big\downarrow \\
h^{q-1}(K) & \xrightarrow{\cup\zeta_p} & H^q(K,\mathbb{Z}/p^{n-1}(q)) & \xrightarrow{p} & H^q(K,\mathbb{Z}/p^n(q)) & \xrightarrow{\text{mod } p} & h^q(K).
\end{array}$$

The top row is exact except at $K_q(K)/p^{n-1}$ and the bottom row is exact. By induction on n, we have only to show the bijectivity of $h_{q,K}\colon k_q(K) \to h^q(K)$ for all q in order to prove Bloch–Kato's theorem.

4.4. Bloch–Kato's Theorem

We review the proof of Bloch–Kato's theorem in the following four steps.

I $\quad \overline{\rho}_m\colon \mathrm{gr}_m k_q(K) \to \mathrm{gr}_m h^q(K)$ is injective for $1 \leqslant m < e'$.

II $\quad \overline{\rho}_0\colon \mathrm{gr}_0 k_q(K) \to \mathrm{gr}_0 h^q(K)$ is injective.

III $\quad h^q(K) = U_0 h^q(K)$ if k is separably closed.

IV $\quad h^q(K) = U_0 h^q(K)$ for general k.

4.4.1. Step I.

Injectivity of $\overline{\rho}_m$ is preserved by taking inductive limit of k. Thus we may assume k is finitely generated over $\mathbb{F}_p$ of transcendence degree $r < \infty$. We also assume $\zeta_p \in K$. Then we get

$$\mathrm{gr}_{e'} h^{r+2}(K) = U_{e'} h^{r+2}(K) \neq 0.$$

For instance, if $r = 0$, then K is a local field and $U_{e'} h^2(K) = {}_p\,\mathrm{Br}(K) = \mathbb{Z}/p$. If $r \geqslant 1$, one can use a cohomological residue to reduce to the case of $r = 0$. For more details see [K1, Sect. 1.4] and [K2, Sect. 3].

For $1 \leqslant m < e'$, consider the following diagram:

$$\begin{array}{ccc}
G_m^q \times G_{e'-m}^{r+2-q} & \xrightarrow{\overline{\rho}_m \times \overline{\rho}_{e'-m}} & \mathrm{gr}_m h^q(K) \oplus \mathrm{gr}_{e'-m} h^{r+2-q}(K) \\
{\scriptstyle \varphi_m}\big\downarrow & & \big\downarrow{\scriptstyle \text{cup product}} \\
\Omega_k^r/d\Omega_k^{r-1} \to G_{e'}^{r+2} & \xrightarrow{\overline{\rho}_{e'}} & \mathrm{gr}_{e'} h^{r+2}(K)
\end{array}$$

where φ_m is, if $p \nmid m$, induced by the wedge product $\Omega_k^{q-1} \times \Omega_k^{r+1-q} \to \Omega_k^r/d\Omega_k^{r-1}$, and if $p \mid m$,

$$\frac{\Omega_k^{q-1}}{Z_1^{q-1}} \oplus \frac{\Omega_k^{q-2}}{Z_1^{q-2}} \times \frac{\Omega_k^{r+1-q}}{Z_1^{r+1-q}} \oplus \frac{\Omega_k^{r-q}}{Z_1^{r-q}} \xrightarrow{\varphi_m} \Omega_k^q/d\Omega_k^{q-1}$$
$$(x_1, x_2, y_1, y_2) \longmapsto x_1 \wedge dy_2 + x_2 \wedge dy_1,$$

and the first horizontal arrow of the bottom row is the projection

$$\Omega_k^q/d\Omega_k^{q-1} \longrightarrow \Omega_k^r/(1 + a\,\mathrm{C})Z_1^r = G_{e'}^{r+2}$$

since $\Omega_k^{r+1} = 0$ and $d\Omega_k^{q-1} \subset (1 + a\,\mathrm{C})Z_1^r$. The diagram is commutative, $\Omega_k^r/d\Omega_k^{r-1}$ is a one-dimensional k^p-vector space and φ_m is a perfect pairing, the arrows in the bottom row are both surjective and $\mathrm{gr}_{e'} h^{r+2}(K) \neq 0$, thus we get the injectivity of $\overline{\rho}_m$.

4.4.2. Step II.

Let K' be a henselian discrete valuation field such that $K \subset K'$, $e(K'|K) = 1$ and $k_{K'} = k(t)$ where t is an indeterminate. Consider

$$\mathrm{gr}_0 h_q(K) \xrightarrow{\cup 1+\pi t} \mathrm{gr}_1 h^{q+1}(K').$$

The right hand side is equal to $\Omega^q_{k(t)}$ by (I). Let ψ be the composite

$$\nu_q(k) \oplus \nu_{q-1}(k) \xrightarrow{\overline{\rho}_0} \mathrm{gr}_0 h^q(K) \xrightarrow{\cup 1+\pi t} \mathrm{gr}_1 h^{q+1}(K') \simeq \Omega^q_{k(t)}.$$

Then

$$\psi\left(\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_q}{x_q}, 0\right) = t\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_q}{x_q},$$
$$\psi\left(0, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{q-1}}{x_{q-1}}\right) = \pm dt \wedge \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{q-1}}{x_{q-1}}.$$

Since t is transcendental over k, ψ is an injection and hence $\overline{\rho}_0$ is also an injection.

4.4.3. Step III.

Denote $sh^q(K) = U_0 h^q(K)$ (the letter s means the symbolic part) and put

$$C(K) = h^q(K)/sh^q(K).$$

Assume $q \geqslant 2$. The purpose of this step is to show $C(K) = 0$. Let $\widetilde{K}$ be a henselian discrete valuation field with algebraically closed residue field $k_{\widetilde{K}}$ such that $K \subset \widetilde{K}$, $k \subset k_{\widetilde{K}}$ and the valuation of K is the induced valuation from $\widetilde{K}$. By Lang [L], $\widetilde{K}$ is a C_1-field in the terminology of [S]. This means that the cohomological dimension of $\widetilde{K}$ is one, hence $C(\widetilde{K}) = 0$. If the restriction $C(K) \to C(\widetilde{K})$ is injective then we get $C(K) = 0$. To prove this, we only have to show the injectivity of the restriction $C(K) \to C(L)$ for any $L = K(b^{1/p})$ such that $b \in \mathcal{O}^*_K$ and $\overline{b} \notin k^p_K$.

We need the following lemmas.

Lemma 1. *Let K and L be as above. Let $G = \mathrm{Gal}(L/K)$ and let $sh^q(L)^G$ (resp. $sh^q(L)_G$) be G-invariants (resp. G-coinvariants). Then*

(i) $sh^q(K) \xrightarrow{\text{res}} sh^q(L)^G \xrightarrow{\text{cor}} sh^q(K)$ *is exact.*

(ii) $sh^q(K) \xrightarrow{\text{res}} sh^q(L)_G \xrightarrow{\text{cor}} sh^q(K)$ *is exact.*

Proof. A nontrivial calculation with symbols, for more details see ([BK, Prop. 5.4]. □

Lemma 2. *Let K and L be as above. The following conditions are equivalent:*

(i) $h^{q-1}(K) \xrightarrow{\text{res}} h^{q-1}(L)_G \xrightarrow{\text{cor}} h^{q-1}(K)$ *is exact.*

(ii) $h^{q-1}(K) \xrightarrow{\cup b} h^q(K) \xrightarrow{\text{res}} h^q(L)$ *is exact.*

Proof. This is a property of the cup product of Galois cohomologies for L/K. For more details see [BK, Lemma 3.2]. □

By induction on q we assume $sh^{q-1}(K) = h^{q-1}(K)$. Consider the following diagram with exact rows:

$$\begin{tikzcd}
 & & h^{q-1}(K) \arrow[d, "\cup b"'] & & \\
0 \arrow[r] & sh^q(K) \arrow[r] \arrow[d, "\mathrm{res}"'] & h^q(K) \arrow[r] \arrow[d, "\mathrm{res}"'] & C(K) \arrow[r] \arrow[d, "\mathrm{res}"'] & 0 \\
0 \arrow[r] & sh^q(L)^G \arrow[r] \arrow[d, "\mathrm{cor}"'] & h^q(L)^G \arrow[r] \arrow[d, "\mathrm{cor}"'] & C(L)^G & \\
0 \arrow[r] & sh^q(K) \arrow[r] & h^q(K). & &
\end{tikzcd}$$

By Lemma 1 (i) the left column is exact. Furthermore, due to the exactness of the sequence of Lemma 1 (ii) and the inductional assumption we have an exact sequence

$$h^{q-1}(K) \xrightarrow{\mathrm{res}} h^{q-1}(L)_G \longrightarrow h^{q-1}(K).$$

So by Lemma 2

$$h^{q-1}(K) \xrightarrow{\cup b} h^q(K) \xrightarrow{\mathrm{res}} h^q(L)$$

is exact. Thus, the upper half of the middle column is exact. Note that the lower half of the middle column is at least a complex because the composite map $\mathrm{cor} \circ \mathrm{res}$ is equal to multiplication by $|L : K| = p$. Chasing the diagram, one can deduce that all elements of the kernel of $C(K) \to C(L)^G$ come from $h^{q-1}(K)$ of the top group of the middle column. Now $h^{q-1}(K) = sh^{q-1}(K)$, and the image of

$$sh^{q-1}(K) \xrightarrow{\cup b} h^q(K)$$

is also included in the symbolic part $sh^q(K)$ in $h^q(K)$. Hence $C(K) \to C(L)^G$ is an injection. The claim is proved.

4.4.4. Step IV.

We use the Hochschild–Serre spectral sequence

$$H^r(G_k, h^q(K_{\mathrm{ur}})) \Longrightarrow h^{q+r}(K).$$

For any q,

$$\Omega^q_{k^{\mathrm{sep}}} \simeq \Omega^q_k \otimes_k k^{\mathrm{sep}}, \qquad Z_1\Omega^q_{k^{\mathrm{sep}}} \simeq Z_1\Omega^q_k \otimes_{k^p} (k^{\mathrm{sep}})^p.$$

Thus, $\mathrm{gr}_m h^q(K_{\mathrm{ur}}) \simeq \mathrm{gr}_m h^q(K) \otimes_{k^p} (k^{\mathrm{sep}})^p$ for $1 \leqslant m < e'$. This is a direct sum of copies of k^{sep}, hence we have

$$H^0(G_k, U_1 h^q(K_{\mathrm{ur}})) \simeq U_1 h^q(K)/U_{e'} h^q(K),$$
$$H^r(G_k, U_1 h^q(K_{\mathrm{ur}})) = 0$$

for $r \geqslant 1$ because $H^r(G_k, k^{\mathrm{sep}}) = 0$ for $r \geqslant 1$. Furthermore, taking cohomologies of the following two exact sequences

$$0 \longrightarrow U_1 h^q(K_{\mathrm{ur}}) \longrightarrow h^q(K_{\mathrm{ur}}) \longrightarrow \nu^q_{k^{\mathrm{sep}}} \oplus \nu^{q-1}_{k^{\mathrm{sep}}} \longrightarrow 0,$$
$$0 \longrightarrow \nu^q_{k^{\mathrm{sep}}} \xrightarrow{\mathrm{C}} Z_1 \Omega^q_{k^{\mathrm{sep}}} \xrightarrow{1-\mathrm{C}^{-1}} \Omega^q_{k^{\mathrm{sep}}} \longrightarrow 0,$$

we have

$$\begin{aligned} H^0(G_k, h^q(K_{\mathrm{ur}})) &\simeq s h^q(K)/U_{e'} h^q(K) \simeq k^q(K)/U_{e'} k^q(K), \\ H^1(G_k, h^q(K_{\mathrm{ur}})) &\simeq H^1(G_k, \nu^q_{k^{\mathrm{sep}}} \oplus \nu^{q-1}_{k^{\mathrm{sep}}}) \\ &\simeq (\Omega^q_k/(1-\mathrm{C}) Z_1 \Omega^q_k) \oplus (\Omega^{q-1}_k/(1-\mathrm{C}) Z_1 \Omega^{q-1}_k), \\ H^r(G_k, h^q(K_{\mathrm{ur}})) &= 0 \end{aligned}$$

for $r \geqslant 2$, since the cohomological p-dimension of G_k is less than or equal to one (cf. [S, II-2.2]). By the above spectral sequence, we have the following exact sequence

$$\begin{aligned} &0 \longrightarrow (\Omega^{q-1}_k/(1-\mathrm{C}) Z_1^{q-1}) \oplus (\Omega^{q-2}_k/(1-\mathrm{C}) Z_1^{q-2}) \longrightarrow h^q(K) \\ &\longrightarrow k_q(K)/U_{e'} k_q(K) \longrightarrow 0. \end{aligned}$$

Multiplication by the residue class of $(1-\zeta_p)^p/\pi^{e'}$ gives an isomorphism

$$\begin{aligned} &(\Omega^{q-1}_k/(1-\mathrm{C}) Z_1^{q-1}) \oplus (\Omega^{q-2}_k/(1-\mathrm{C}) Z_1^{q-2}) \\ &\longrightarrow (\Omega^{q-1}_k/(1+a\,\mathrm{C}) Z_1^{q-1}) \oplus (\Omega^{q-2}_k/(1+a\,\mathrm{C}) Z_1^{q-2}) = \mathrm{gr}_{e'} k_q(K), \end{aligned}$$

hence we get $h^q(K) \simeq k_q(K)$.

References

[BK] S. Bloch and K. Kato, p-adic étale cohomology, Publ. Math. IHES 63(1986), 107–152.

[BT] H. Bass, H. and J. Tate, The Milnor ring of a global field, In Algebraic K-theory II, Lect. Notes in Math. 342, Springer-Verlag, Berlin, 1973, 349–446.

[F] I. Fesenko, Class field theory of multidimensional local fields of characteristic 0, with the residue field of positive characteristic, Algebra i Analiz (1991); English translation in St. Petersburg Math. J. 3(1992), 649–678.

[FV] I. Fesenko and S. Vostokov, Local Fields and Their Extensions, AMS, Providence RI, 1993.

[K1] K. Kato, A generalization of local class field theory by using K-groups. II, J. Fac. Sci. Univ. Tokyo 27(1980), 603–683.

[K2] K. Kato, Galois cohomology of complete discrete valuation fields, In Algebraic K-theory, Lect. Notes in Math. 967, Springer-Verlag, Berlin, 1982, 215–238.

[L] S. Lang, On quasi-algebraic closure, Ann. of Math. 55(1952), 373–390.

[MS1] A. Merkur'ev and A. Suslin, K-cohomology of Severi-Brauer varieties and the norm residue homomorphism, Math. USSR Izvest. 21(1983), 307–340.

[MS2] A. Merkur'ev and A. Suslin, Norm residue homomorphism of degree three, Math. USSR Izvest. 36(1991), 349–367.

[R] M. Rost, Hilbert 90 for K_3 for degree-two extensions, preprint, 1986.

[S] J.-P. Serre, Cohomologie Galoisienne, Lect. Notes in Math. 5, Springer-Verlag, 1965.

[T] J. Tate, Relations between K_2 and Galois cohomology, Invent. Math. 36(1976), 257–274.

[V] V. Voevodsky, The Milnor conjecture, preprint, 1996.

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